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On weakly $\mathfrak{Z}$-permutable subgroups of finite groups II

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Abstract Let G be a finite group. We say that $\mathfrak{Z}$ is a complete set of Sylow subgroups of G if for each prime p dividing the order of G, $\mathfrak{Z}$ contains exactly one Sylow p-subgroup of G, G_p say. A subgroup of G is said to be $\mathfrak{Z}$-permutable in G if it permutes with every member of $\mathfrak{Z}$. A subgroup H of G is said to be weakly $\mathfrak{Z}$-permutable in G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\mathfrak{Z}}$, where $H_{\mathfrak{Z}}$ is the subgroup of H generated by all those subgroups of H which are $\mathfrak{Z}$-permutable in G . In this paper, we prove that G is supersolvable if the maximal subgroups of $G_p \cap F^*(G)$ are weakly $\mathfrak{Z}$-permutable in G, for every $G_p \in \mathfrak{Z}$, where $F^*(G)$ is the generalized Fitting subgroup of G. Also, we prove that if $\mathfrak{F}$ is a saturated formation containing the class of all supersolvable groups, then $G \in \mathfrak{F}$ if and only if there is a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{Z}$-permutable in G, for every $G_p \in \mathfrak{Z}$.

Mathematics Subject Classification 20D10 · 20D15 · 20D20 · 20F16

الملخص

لتكن G زمرة منتهية. نقول إن 3 مجموعة تامة من زمر سيلو الجزئية من G إذا احتوت 3 لكل قاسم أولي p من قواسم مرتبة G على زمرة سيلو جزئية واحدة بالضبط من G، ولتكن G_p. تسمى زمرة جزئية من G زمرة 3-قابلة للتبديل في G إذا كانت إبدالية مع كل عنصر من عناصر 3. تسمى الزمرة الجزئية H من G زمرة 3-قابلة للتبديل بشكل ضعيف في G إذا وجدت زمرة جزئية K ناظمية جزئيا من G بحيث يكون $G = HK$ وَ $H \cap K \leq H_3$، حيث H_3 الزمرة الجزئية من H المولدة بواسطة كل الزمر الجزئية من H التي تكون 3-قابلة للإبدال في G. في هذه الورقة، نثبت أن G فوق حلولة إذا كانت الزمر الجزئية الأعظمية من $G_p \cap F^*(G)$ من الزمر 3-القابلة للإبدال بشكل ضعيف في G لكل $G_p \in 3$، حيث $F^*(G)$ زمرة فتنغ الجزئية المعممة من G. نثبت أيضا أنه إذا كان $\mathfrak{F}$ تكويناً مشبعاً يحتوي صف الزمر فوق الحلولة، فإن $G \in \mathfrak{F}$ إذا وجدت وفقط إذا وجدت زمرة جزئية ناظمية H من G بحيث تكون $G/H \in \mathfrak{F}$ وكانت الزمر الجزئية الأعظمية من $G_p \cap F^*(G)$ هي 3-قابلة للإبدال بشكل ضعيف في G لكل $G_p \in 3$.

1 Introduction and statement of results

All groups considered in the sequel will be finite. Most of the notation is standard and can be found in Ballester-Bolinches et al. [2] and Doerk and Hawkes [3]. In addition, $\pi(G)$ denotes the set of distinct primes dividing $|G|$ and G_p is a Sylow p-subgroup of the group G for some prime $p \in \pi(G)$.

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Recall that a subgroup H of a group G is said to be S-permutable in G if it permutes with every Sylow subgroup of G. This concept was introduced by Kegel [7], who called these subgroups S-quasinormal, and has been studied extensively by many authors. Asaad and Heliel [1] generalized S-permutability property by requiring permutability only with the members of a complete set of Sylow subgroups. We say that $\mathfrak{Z}$ is a complete set of Sylow subgroups of G if for each prime p dividing $|G|$, $\mathfrak{Z}$ contains exactly one Sylow p-subgroup of G. A subgroup H of G is said to be $\mathfrak{Z}$-permutable in G if H permutes with every member of $\mathfrak{Z}$. Following Wang [11], we say that a subgroup H of a group G is c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = \mathrm{Core}_G(H)$ is the largest normal subgroup of G contained in H. One can easily find groups with $\mathfrak{Z}$-permutable subgroups that are not c-normal and conversely there are also groups with c-normal subgroups that are not $\mathfrak{Z}$-permutable subgroups; see Examples 1, 2 and 3 in Heliel et al. [4]. In fact, there is no inclusion relationship between c-normality and $\mathfrak{Z}$-permutability. Consequently the authors in [4] unified and generalized both of $\mathfrak{Z}$-permutability and c-normality concepts as follows: a subgroup H of a group G is said to be weakly $\mathfrak{Z}$-permutable in G if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{\mathfrak{Z}}$, where $H_{\mathfrak{Z}}$ is the subgroup of H generated by all those subgroups of H which are $\mathfrak{Z}$-permutable in G. Using this new subgroup embedding property, the authors in [4] studied the structure of a group G when all maximal subgroups of certain or every member of a complete set of Sylow subgroups of some normal subgroup of G are weakly $\mathfrak{Z}$-permutable in G. There they achieved results unified and generalized several recent results in the literature. The present paper may be viewed as a continuation of [4]; to be more precise, the following results have been proved in [4]:

Theorem 1.1 *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and let p be the smallest prime dividing $|G|$. If the maximal subgroups of $G_p \in \mathfrak{Z}$ are weakly $\mathfrak{Z}$-permutable in G, then G is p-nilpotent.*

Theorem 1.2 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Then the following are equivalent:*

(a) $G \in \mathfrak{F}$.
(b) *There is a solvable normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of the Fitting subgroup $F(H)$ are weakly $\mathfrak{Z}$-permutable in G.*

The main object here is to take the above-mentioned results further by proving:

Theorem 1.3 *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. If the maximal subgroups of $G_p \cap F^*(G)$ are weakly $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$, then G is supersolvable.*

Theorem 1.4 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Then the following are equivalent:*

(a) $G \in \mathfrak{F}$.
(b) *There is a normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$.*

Theorems 1.3 and 1.4 improve and extend some well-known results in the literature; see Corollaries 3.1, 3.2, 3.3, 3.4 and 3.5.

Recall that for any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G. $F^*(G)$ is an important characteristic subgroup of G and it is a natural generalization of the Fitting subgroup $F(G)$. By [5, X 13], $F^*(G) \neq 1$ if $G \neq 1$. The basic properties of $F^*(G)$ can be found in [5, X 13]. The reader is referred to [3] for basic properties and results about saturated formations.

2 Known results

Lemma 2.1 *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. If the maximal subgroups of G_p are weakly $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$, then G is supersolvable.*

Proof See [4, Corollary 3.4]. □

Lemma 2.2 *Let G be a group. Then:*

(a) *$F^*(G) = F(G)E(G)$ and $[F(G), E(G)] = 1$, where $E(G)$ is the layer subgroup of G.*

(b) *$E(G)$ is a perfect quasinilpotent characteristic subgroup of G.*
(c) *If N is a normal subgroup of G, then $F^*(N) \leq F^*(G)$.*
(d) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
(e) *Suppose that P is a normal subgroup of G contained in $O_p(G)$ for some prime p, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*
(f) *Suppose that K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Proof For (a), (b) and (c), see [5, p. 128, Definition 13.14 and p. 127, Corollary 13.11(c)]. For the first part of (d), see [5, p. 127, Corollary 13.11(a)], and the rest comes from the fact that $F^*(G) = F(G)E(G)$ by (a) and in case $F^*(G)$ is solvable, we have that $E(G)$ is a perfect solvable group by (b) which implies that $E(G) = 1$ and hence $F^*(G) = F(G)$.

For (e) and (f), see [10, Lemma 2.3(6) and (7)]. □

Lemma 2.3 *Let H and N be subgroups of a group G such that N is normal in G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Then:*

(a) *If $H \leq N$ and H is weakly $\mathfrak{Z}$-permutable in G, then H is weakly $\mathfrak{Z} \cap N$-permutable in N.*
(b) *If $N \leq H$, then H is weakly $\mathfrak{Z}$-permutable in G if and only if H/N is weakly $\mathfrak{Z}N/N$-permutable in G/N.*
(c) *If $(|H|, |N|) = 1$ and H is weakly $\mathfrak{Z}$-permutable in G, then HN/N is weakly $\mathfrak{Z}N/N$-permutable in G/N.*
(d) *If H is a p-subgroup of G for some prime p such that H is weakly $\mathfrak{Z}$-permutable in G but it is not $\mathfrak{Z}$-permutable in G, then there exists a normal subgroup M of G with $|G : M| = p$ and $G = HM$.*

Proof See [4, Lemma 2.3]. □

Lemma 2.4 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Then the following are equivalent:*

(a) *$G \in \mathfrak{F}$.*
(b) *There is a solvable normal subgroup H in G such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are weakly $\mathfrak{Z}$-permutable in G.*

Proof See [4, Theorem 1.6]. □

Lemma 2.5 *Let N be a normal subgroup of a group G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Suppose that P is a p-subgroup of G for some prime p. Then:*

(a) *If M/N is a maximal subgroup of PN/N, then $M = (M \cap P)N$, where $M \cap P$ is a maximal subgroup of P.*
(b) *If $(|P|, |N|) = 1$ and the maximal subgroups of P are weakly $\mathfrak{Z}$-permutable in G, then the maximal subgroups of PN/N are weakly $\mathfrak{Z}N/N$-permutable in G/N.*

Proof See [4, Lemma 2.4]. □

Lemma 2.6 *Let N be a normal nilpotent subgroup of a group G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Suppose that P is a normal p-subgroup of G, for some prime p, with $P \leq N$. If the maximal subgroups of the Sylow subgroups of N are weakly $\mathfrak{Z}$-permutable in G, then the maximal subgroups of the Sylow subgroups of N/P are weakly $\mathfrak{Z}P/P$-permutable in G/P. In other words, if the maximal subgroups of the members of $\mathfrak{Z} \cap N$ are weakly $\mathfrak{Z}$-permutable in G, then the maximal subgroups of the members of $(\mathfrak{Z}P/P) \cap (N/P)$ are weakly $\mathfrak{Z}P/P$-permutable in G/P.*

Proof Let Q be a Sylow q-subgroup of N. Since Q is characteristic in N and $N \trianglelefteq G$, we have $Q \trianglelefteq G$. Assume that $q \neq p$. As $(|Q|, |P|) = 1$ and the maximal subgroups of Q are weakly $\mathfrak{Z}$-permutable in G, we have, by Lemma 2.5(b), that the maximal subgroups of QP/P are weakly $\mathfrak{Z}P/P$-permutable in G/P. Hence, we may assume that $q = p$ and so $P \leq Q$. Let M/P be a maximal subgroup of Q/P. By Lemma 2.5(a), M is a maximal subgroup of Q. The hypothesis and Lemma 2.3(b) imply that M/P is weakly $\mathfrak{Z}P/P$-permutable in G/P. Therefore, the maximal subgroups of Q/P are weakly $\mathfrak{Z}P/P$-permutable in G/P. Thus, the maximal subgroups of the Sylow subgroups of N/P are weakly $\mathfrak{Z}P/P$-permutable in G/P. Clearly, $\mathfrak{Z} \cap N$ is the set of all Sylow subgroups of N as N is nilpotent. Since $\mathfrak{Z}P/P$ is a complete set of Sylow subgroups of G/P and N/P is a normal nilpotent subgroup of G/P, it follows that $(\mathfrak{Z}P/P) \cap (N/P)$ is the set of all Sylow subgroups of N/P. Thus, the maximal subgroups of $(\mathfrak{Z}P/P) \cap (N/P)$ are weakly $\mathfrak{Z}P/P$-permutable in G/P. □

Lemma 2.7 *Let H be a subgroup of a group G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Then $H_{\mathfrak{Z}}$ is $\mathfrak{Z}$-permutable in G and $H_G \leq H_{\mathfrak{Z}}$.*

Proof See [4, Lemma 2.2(a)]. □

Lemma 2.8 *Let H and N be subgroups of a group G such that N is normal in G, and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. If H is $\mathfrak{Z}$-permutable in G, then $H \cap N$ is $\mathfrak{Z}$-permutable in G.*

Proof See [4, Lemma 2.1(e)]. □

Lemma 2.9 *Let G be a group. Assume that N is a normal subgroup of G $(N \neq 1)$ and $N \cap \Phi(G) = 1$. Then the Fitting subgroup $F(N)$ of N is the direct product of the minimal normal subgroups of G which are contained in $F(N)$.*

Proof See [9, Lemma 2.6]. □

3 Proofs

Proof of Theorem 1.3. Assume that the result is false and let G be a counterexample of minimal order. Then the following statements about G are true:

(1) $F^*(G) \neq G$.

Suppose that $F^*(G) = G$. Since the maximal subgroups of G_p are weakly $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$, it follows, by Lemma 2.1, that G is supersolvable, a contradiction. Thus, $F^*(G) \neq G$.

(2) Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

By (1), $F^*(G) \neq G$ and so there exists a proper normal subgroup N of G containing $F^*(G)$. Since $F^*(G) \trianglelefteq N \trianglelefteq G$, it follows, by Lemma 2.2(c), that $F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$. By Lemma 2.2(d), $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$ and hence $F^*(G) = F^*(N)$. Lemma 2.3(a) implies that the maximal subgroups of $G_p \cap F^*(N)$ are weakly $\mathfrak{Z} \cap N$-permutable in N, for all $G_p \cap N \in \mathfrak{Z} \cap N$. Therefore, N is supersolvable by the minimal choice of G. Thus, every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

(3) $F^*(G) = F(G)$.

Since $F^*(G) \neq G$ by (1), it follows that $F^*(G)$ is supersolvable by (2). Thus, $F^*(G) = F(G)$ by Lemma 2.2(d).

(4) G has no normal subgroup of prime order and, therefore, $Z(G) = 1$.

Let K be a normal subgroup of G of prime order. Assume that $C_G(K)$ is a proper subgroup of G. By [3, p. 36, Theorem 10.6(b)] and (3), $F^*(G) = F(G) \leq C_G(K)$. Therefore, $C_G(K)$ is supersolvable by (2) and, since $G/C_G(K)$ is isomorphic to a subgroup of the cyclic group $Aut(K)$, it follows that G is solvable. Thus, G is supersolvable by Lemma 2.4, a contradiction. Consequently we may assume that $C_G(K) = G$ and so $K \leq Z(G)$. Lemma 2.2(f) and (3) imply that $F^*(G/K) = F^*(G)/K = F(G)/K$. By Lemma 2.6, the maximal subgroups of the Sylow subgroups of $F^*(G/K) = F(G)/K$ are weakly $\mathfrak{Z}K/K$-permutable in G/K. Therefore, G/K satisfies the hypothesis of the theorem and hence G/K is supersolvable by the minimal choice of G. This implies that G is supersolvable as K has a prime order, a contradiction. Thus, G has no normal subgroup of prime order and it follows easily that $Z(G) = 1$.

Let P be a Sylow p-subgroup of $F(G)$, then the following holds:

(5) $\Phi(P) = 1$ and, therefore, $P$is abelian.

Suppose that $\Phi(P) \neq 1$. Clearly, as $\Phi(P)$ is characteristic in P and $P \trianglelefteq G$, $\Phi(P) \trianglelefteq G$. By Lemma 2.2(e) and (3), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P) = F(G)/\Phi(P)$. Lemma 2.6 implies that the maximal subgroups of the Sylow subgroups of $F^*(G/\Phi(P)) = F(G)/\Phi(P)$ are weakly $\mathfrak{Z}\Phi(P)/\Phi(P)$-permutable in $G/\Phi(P)$. Therefore, $G/\Phi(P)$ is supersolvable by the minimal choice of G. Since $\Phi(P) \leq \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable. But since the class of supersolvable groups is a saturated formation, then G is supersolvable, a contradiction. Thus, $\Phi(P) = 1$ and it follows that P is abelian.

(6) $G = PO^p(G)$.

Suppose that $G \neq PO^p(G)$. Since $F^*(G) = F(G) \leq PO^p(G)$ by (3), it follows that $PO^p(G)$ is supersolvable by (2) and hence $O^p(G)$ is supersolvable. Therefore, G is solvable as $G/O^p(G)$ is a p-group. By Lemma 2.4, G is supersolvable, a contradiction. Thus, $G = PO^p(G)$.

(7) $U \cap O^p(G)$ is a normal subgroup of G for any maximal subgroup U of P.

Let x be an element of G and let $V = U^{x^{-1}}$. It is clear that V is a maximal subgroup of P as P is normal in G and $|V| = |U|$. By hypothesis, V is weakly $\mathfrak{Z}$-permutable in G. Then there exists a subnormal subgroup K of G such that $G = VK$ and $V \cap K \leq V_{\mathfrak{Z}}$. Since P is abelian by (5) and $G = PK$, it follows that $P \cap K$ is a normal subgroup of G. Clearly, $O^p(G) \leq K$ as K is subnormal in G and $|G : K|$ is a power of p. By (6), $K = PO^p(G) \cap K = (P \cap K)O^p(G)$ and hence K is a normal subgroup of G. Therefore, $V_{\mathfrak{Z}}K$ is a subgroup of G. Obviously, $V \cap V_{\mathfrak{Z}}K = V_{\mathfrak{Z}}(V \cap K) = V_{\mathfrak{Z}}$, and hence $V \cap V_{\mathfrak{Z}}K$ is $\mathfrak{Z}$-permutable in G by Lemma 2.7. Therefore, $V \cap O^p(G) = (V \cap V_{\mathfrak{Z}}K) \cap O^p(G)$ is $\mathfrak{Z}$-permutable in G by Lemma 2.8. Thus, $(V \cap O^p(G))G_q = (U^{x^{-1}} \cap O^p(G))G_q$ is a subgroup of G, for all $x \in G$ and for all $G_q \in \mathfrak{Z}$. This implies that $(U \cap O^p(G))G_q^x = ((U^{x^{-1}})^x \cap O^p(G)^x)G_q^x = (U^{x^{-1}} \cap O^p(G))^x G_q^x = ((U^{x^{-1}} \cap O^p(G))G_q)^x$ is a subgroup of G, for all $x \in G$ and for all $G_q \in \mathfrak{Z}$. As the Sylow subgroups of G are conjugate, we have that $U \cap O^p(G)$ is an S-permutable in G. By [2, p. 17, Lemma 1.2.16], $O^p(G) \leq N_G(U \cap O^p(G))$. Also, $U \cap O^p(G)$ is a normal subgroup of P as P is abelian by (5). Consequently, $G = PO^p(G) \leq N_G(U \cap O^p(G))$ by (6) and hence $U \cap O^p(G)$ is normal in G. Thus, $U \cap O^p(G)$ is a normal subgroup of G for any maximal subgroup U of P.

(8) $P \cap O^p(G) \neq 1$.

Suppose that $P \cap O^p(G) = 1$. Since $[P, O^p(G)] \leq P \cap O^p(G) = 1$, it follows that $O^p(G) \leq C_G(P)$. Also, $P \leq C_G(P)$ as P is abelian by (5). Therefore, $G = PO^p(G) \leq C_G(P)$ by (6) and so $P \leq Z(G)$ which is a contradiction with (4). Thus, $P \cap O^p(G) \neq 1$.

(9) $P \cap \Phi(G) \neq 1$.

Suppose that $P \cap \Phi(G) = 1$. By Lemma 2.9, $P = L_1 \times L_2 \times \cdots \times L_t$, where each L_i is a minimal normal subgroup of G, for $i = 1, 2, \ldots, t$. Since $P \cap O^p(G) \neq 1$ by (8) and $\Phi(P) = 1$ by (5), then there exists a maximal subgroup U of P such that $P = (P \cap O^p(G))U$. This implies that $|(P \cap O^p(G)) : (U \cap O^p(G))| = |P : U| = p$. By (7), $U \cap O^p(G)$ is a normal subgroup of G. Therefore, $(P \cap O^p(G))/(U \cap O^p(G))$ is a chief factor of G with order p. Consider P as an operator group with operator domain $\Omega = Inn(G)$. Then the series $1 \trianglelefteq L_1 \trianglelefteq L_1L_2 \trianglelefteq \cdots \trianglelefteq L_1L_2\cdots L_t = P$ is an Ω-composition series of the Ω-group P. Since $(P \cap O^p(G))/(U \cap O^p(G))$ is an Ω-composition factor of P, it follows, by the Jordan–Hölder Theorem, that $(P \cap O^p(G))/(U \cap O^p(G)) \cong L_i$, for some $i (1 \leq i \leq t)$. Therefore, $|L_i| = p$, for some $i (1 \leq i \leq t)$, a contradiction with (4). Thus, $P \cap \Phi(G) \neq 1$.

Let L be a minimal subgroup of G contained in $P \cap \Phi(G)$, then the following holds:

(10) $F(G/L) = F(G)/L$.

Let $K/L = F(G/L)$. Then K is a normal nilpotent subgroup of G by [6, p. 270, Satz 3.5] and so $K \leq F(G)$. Thus, $F(G/L) = F(G)/L$.

(11) L is the unique minimal normal subgroup of G contained in P.

By Lemma 2.2(a) and (10), we have that $F^*(G/L) = F(G/L)E(G/L) = F(G)/L\ S/L$ and $[F(G)/L, S/L] = 1$, where $S/L = E(G/L)$ is the layer subgroup of G/L. Since $[F(G)/L, S/L] = 1$, then $[F(G), S] \leq L$. Let K be a minimal normal subgroup of G contained in P distinct from L. Then $[K, S] \leq K \cap L = 1$ and hence $S \leq C_G(K)$. By (4), $C_G(K)$ is a proper subgroup of G. Since $F^*(G) = F(G) \leq C_G(K)$ by (3) and [3, p. 36, Theorem 10.6(b)], we have that $C_G(K)$ is supersolvable by (2) and so S is solvable. Since S/L is a solvable perfect group by Lemma 2.2(b), then $S/L = 1$ and it follows that $F^*(G/L) = F(G)/L$. By Lemma 2.6, the maximal subgroups of the Sylow subgroups of $F^*(G/L) = F(G)/L$ are weakly $\mathfrak{Z}L/L$-permutable in G/L. Thus, G/L satisfies the hypothesis of the theorem and hence G/L is supersolvable by the minimal choice of G. Since $L \leq \Phi(G)$, it follows that $G/\Phi(G)$ is supersolvable. This implies that G is supersolvable as the class of supersolvable groups is a saturated formation, a contradiction. Thus, L is the unique minimal normal subgroup of G contained in P.

(12) The final contradiction.

By (5), there exists a maximal subgroup U of P such that $P = LU$. If $L \leq U \cap O^p(G)$, then $L \leq U$ and hence $P = U$, a contradiction. Therefore, L is not contained in $U \cap O^p(G)$. Since $U \cap O^p(G)$ is a normal subgroup of G by (7) and $L \nleq U \cap O^p(G)$, it follows, by (11), that $U \cap O^p(G) = 1$. Clearly, $P = (P \cap O^p(G))U$ as $L \leq P \cap O^p(G)$ by (8) and (11). Thus, $|P \cap O^p(G)| = |P : U| = p$, a contradiction with (4). This completes the proof of the theorem. □

Proof of Theorem 1.4. (a) $\Rightarrow$ (b). If $G \in \mathfrak{F}$, then (b) is true with $H = 1$.

(b) $\Rightarrow$ (a). By Lemma 2.3(a), the maximal subgroups of $G_p \cap F^*(H)$ are weakly $\mathfrak{Z} \cap H$-permutable in H, for all $G_p \cap H \in \mathfrak{Z} \cap H$. Theorem 1.3 implies that H is supersolvable and hence $F^*(H) = F(H)$. Therefore, H is a solvable normal subgroup of G with $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are weakly $\mathfrak{Z}$-permutable in G. By Lemma 2.4, $G \in \mathfrak{F}$. This completes the proof of the theorem. □

The following well-known results in the literature are immediate consequences of Theorems 1.3 and 1.4.

Corollary 3.1 [8, Theorem 3.1] *Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G and let H be a normal subgroup of G such that $G/H \in \mathfrak{U}$. If the maximal subgroups of $G_p \cap F^*(H)$ are $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$, then $G \in \mathfrak{U}$.*

Corollary 3.2 [9, Theorem 3.1] *Let G be a group with a normal subgroup H such that $G/H \in \mathfrak{U}$. If the maximal subgroups of the Sylow subgroups of $F^*(H)$ are S-permutable in G, then $G \in \mathfrak{U}$.*

Corollary 3.3 [8, Main Theorem] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are $\mathfrak{Z}$-permutable in G, for all $G_p \in \mathfrak{Z}$.*

Corollary 3.4 [9, Theorem 3.4] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let G be a group. If G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F^*(H)$ are S-permutable in G, then $G \in \mathfrak{F}$.*

Corollary 3.5 [10, Theorem 3.1] *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups $\mathfrak{U}$ and let G be a group. If G has a normal subgroup H such that $G/H \in \mathfrak{F}$ and the maximal subgroups of the Sylow subgroups of $F^*(H)$ are c-normal in G, then $G \in \mathfrak{F}$.*

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